Relation and Mapping

1. (a).
Let $f: S \to T$ and $g: T \to U$ be two mapping such that $g \circ f$ is bijective. prove that f is injective and g is surjective.

$\Rightarrow$ If possible, let f be not injective. Then $\exists$ two distinct elements x_1, x_2 in S such that $f(x_1) = f(x_2)$.

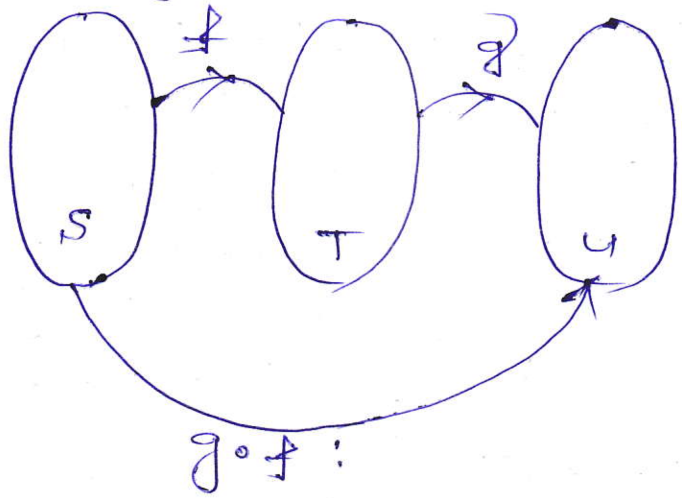
So, $g \circ f(x_1) = g \circ f(x_2)$ and this Contradicts the fact that $g \circ f$ is injective.

$\therefore$ f is injective.

Since $g \circ f$ is bijective, $g \circ f$ is injective as well as $g \circ f$ is surjective.

Let y be an element of U. Since $g \circ f$ is surjective, $\exists$ an element x in S, such that $g \circ f(x) = y$. i.e $gf(x) = y$

This shows that y has a pre-image $f(x)$ in T under the mapping g.
Since y is arbitrary, g is surjective.



b) Prove that any partition of a non empty set S induces an equivalence relation on S.

⇒ Let, there be a partition P of the set S into the subsets.

Let us define a relation ρ on the set S to mean that $a \rho b$ holds, if a, b belong to one and the same subset of the partition P.

Let $a \in S$. Then $a \rho a$ holds since a and a belong to one and the same subset of the partition P. Therefore ρ is reflexive.

Let $a, b \in S$ and $a \rho b$. Then a and b belong to the one and the same subset of the partition P and therefore b and a belong to the same subset of P. That is $a \rho b \Rightarrow b \rho a$. $\therefore \rho$ is symmetric.

Let $a, b, c \in S$ and $a \rho b$, $b \rho c$ both hold. Then a and b belong to the same subset, say S' of P; b and c belong to one and the same subset S'' of P.

Now S' and S'' being subsets of a partition, they must be either identical or disjoint. Since $b \in S'$, $b \in S''$, it follows that S' and S'' are the same subset of the partition P and consequently a and c belong to the one and the same subset of P, i.e $a \rho b$ and $b \rho c \Rightarrow a \rho c$

$\therefore \rho$ is Transitive.

Thus ρ is an equivalent relation on S.

This completes the proof.

1.(c) Let $S = \{x \in \mathbb{R} : -1 < x < 1\}$. Find a bijective mapping from $\mathbb{R}$ onto S. Also find its inverse.

$\Rightarrow$ Let us consider the mapping $f: \mathbb{R} \to S$ defined by $f(x) = \dfrac{x}{1+|x|}$, $x \in \mathbb{R}$

Let $x > 0$. Then $f(x) = \dfrac{x}{1+x}$ and $0 < f(x) < 1$.

Let $x = 0$. Then $f(x) = 0$.

Let $x < 0$. Then $f(x) = \dfrac{x}{1-x} = \dfrac{1}{1-x} - 1$

and $-1 < f(x) < 0$.

Let $x_1, x_2 \in \mathbb{R}$ and $f(x_1) = f(x_2)$. Then

$$\frac{x_1}{1+|x_1|} = \frac{x_2}{1+|x_2|}.$$

$\Rightarrow$ x_1 and x_2 are either both positive or both negative.

Let $x_1 > 0$ and $x_2 > 0$.

Then $f(x_1) = f(x_2) \Rightarrow \dfrac{x_1}{1+x_1} = \dfrac{x_2}{1+x_2} \Rightarrow x_1 = x_2$

Let $x_1 < 0$ and $x_2 < 0$

Then $f(x_1) = f(x_2) \Rightarrow \dfrac{x_1}{1-x_1} = \dfrac{x_2}{1-x_2} \Rightarrow x_1 = x_2$

It follows that $x_1 \neq x_2$ in $\mathbb{R} \Rightarrow f(x_1) \neq f(x_2)$

So f is injective.

Let y be an element in S and let $0 < y < 1$. Let us examine if $x(>0)$ in $\mathbb{R}$ be a pre-image of y. Then $\dfrac{x}{1+x} = y$.

or $x = \dfrac{y}{1-y} \in \mathbb{R}$, since $y \in \mathbb{R}$.

$\therefore$ $\dfrac{y}{1-y}$ is a pre-image of y.

Let y be an element in S and let $-1 < y < 0$. Let us examine if $x(<0)$ in $\mathbb{R}$ be a pre-image of y. Then $\dfrac{x}{1-x} = y$

or, $x = \frac{y}{1+y} \in \mathbb{R}$, since $y \in \mathbb{R}$, $\therefore \frac{y}{1+y}$ is a pre-image of y.

Let $y = 0 \in S$. Then $x = 0$ is a pre-image of y.

It follows that each y in S has a pre-image in $\mathbb{R}$. So f is surjective.

Since f is injective as well as surjective, f is a bijection.

Since f is a bijection, each y in S has a unique pre-image.

For $y > 0$, the pre-image is $\frac{y}{1-y} = \frac{y}{1-|y|}$

For $y < 0$, the pre-image is $\frac{y}{1+y} = \frac{y}{1-|y|}$

For $y = 0$, the pre-image is $0 = \frac{y}{1-|y|}$.

It follows that for each y in S the pre-image is $\frac{y}{1-|y|}$.

Hence $f^{-1} : S \to \mathbb{R}$ is defined by

$$f^{-1}(x) = \frac{x}{1-|x|}, \quad x \in S.$$

2. (a). Give an example to illustrate that union of two equivalence relations need not be an equivalence relations.

$\Rightarrow$ Consider two equivalence relations on A given by,

$$R = \{(1,1), (2,2), (3,3), (2,3), (3,2)\};$$
$$P = \{(1,1), (2,2), (3,3), (3,1), (1,3)\}.$$

Here $R \cup P = \{(1,1), (2,2), (3,3), (2,3), (3,2), (3,1), (1,3)\}$, which is not an equivalence relation as its lacks transitivity indeed $(2,3), (3,1) \in R \cup P$ but $(2,1) \notin R \cup P$.

2. (b). Suppose for two functions f and g such that $g \circ f$ is surjective. Is f necessarily surjective? Justify your answer.

⇒ No:

for example, let $f: \mathbb{Z} \times \mathbb{Z}$ be defined by $f(x) = 2x$, $x \in \mathbb{Z}$ and $g: \mathbb{Z} \to \mathbb{Z}$ be defined by $g(x) = \left[\frac{x}{2}\right]$, $x \in \mathbb{Z}$. $[x]$ denotes the greatest integer $\leq x$. Then $g \circ f: \mathbb{Z} \to \mathbb{Z}$ is defined by $g \circ f(x) = x$, $x \in \mathbb{Z}$

$g \circ f = i_{\mathbb{Z}}$ and is, therefore, surjective; but f is not surjective.

3. (a) Suppose f and g are two mappings such that $f \circ g$ is defined. Prove that if

i) $g \circ f$ is injective then f is injective.

ii) If $g \circ f$ is surjective, then g is surjective.

⇒

i) If possible, let f be not injective. Then $\exists\, a_1, a_2 \in A$ such that $a_1 \neq a_2$; but $f(a_1) = f(a_2)$. $\Rightarrow g(f(a_1)) = g(f(a_2))$ as g is well defined; i.e $(g \circ f)(a_1) = (g \circ f)(a_2)$ which contradicts the injectivity of $g \circ f$. Hence if $g \circ f$ is injective, so must be f.

ii) Let $f: A \to B$ and $g: B \to C$.

Let $c \in C$. since $g \circ f: A \to C$ is surjective; there exist some $a \in A$ such that $(g \circ f)(a) = c$. i.e $g(f(a)) = c$. ⇒ for $c \in C$, there exist

$f(a) \in B$ such that $f(a)$ is a pre-image of c under g. Since c is an arbitrary element of C, we conclude that g is surjective.

4. (a). Construct an equivelence relation on the set $\{1, 2, 3\}$.

$\Rightarrow$ Let $A = \{1, 2, 3\}$

Let us consider the relation R defined on the set A by

$$R = \{(1,1), (2,2), (3,3), (2,3), (3,2)\}$$

now for all $a \in A \Rightarrow (a, a) \in R$.

$\therefore$ R is reflexive.

$a R b \Rightarrow b R a \quad \forall a, b \in A$

$\therefore$ R is symmetric.

$a R b$ and $b R c$

$\Rightarrow a R c, \quad \forall a, b, c \in A$.

$\therefore$ R is transitive.

Since R is reflexive, transitive and symmetric, then R is a equivalence relation defined on the set A.

4. (b) Show that the map $g: R \to R$ defined by $g(x) = x^{16}$ is neither injective nor surjective, R being the set of all reals.

$\Rightarrow$ $g: R \to R$.

$g(x) = x^{16} \quad \forall x \in R$

$g(1) = (1)^{16} = 1$; $g(-1)^{16} = (-1)^{16} = 1$

$\therefore 1 \neq -1$ but $g(1) = g(-1)$

$\therefore g(x) = x^{16}$ is not injective.

Negative numbers in the codomain set of g has no pre-image in the domain of g.

$\therefore g(x) = x^{16}$ is not surjective.

5. a) Construct a map $f: N \to N$ which is not injective but surjective, and also construct a map $g: N \to N$ which is not surjective but injective. Can you construct similar maps on replacing N by a finite set? Justify your answer. (N being the set of all natural numbers).

$\Rightarrow$ $f: N \to N$ defined by $f(n) = \left[\frac{n+1}{2}\right], n \in N$ is surjective but not injective.

$f(1) = \left[\frac{1+1}{2}\right] = [1] = 1$

$f(2) = \left[\frac{2+1}{2}\right] = [1.5] = 1$

$\because 1 \neq 2$ but $f(1) = f(2)$

$\therefore f$ is not injective.

(i) $g: N \to N$ be defined by $g(n) = n+1, n \in N$

Let $n_1, n_2 \in N$ (domain set) and $n_1 \neq n_2$.

now, $f(n_1) - f(n_2)$

$= (n_1 + 1) - (n_2 + 1)$

$= n_1 - n_2 \neq 0 \quad [\because n_1 \neq n_2]$

$\therefore f(n_1) \neq f(n_2) \quad \forall n_1, n_2 \in N$

i.e. $n_1 \neq n_2 \Rightarrow f(n_1) \neq f(n_2)$

$\therefore f$ is injective.

Let $y \in N$ (co domain set)

Then $g(x) = y \Rightarrow n+1 = y$

$\Rightarrow n = y - 1$

Let $y = 1 \in N$

Then $n = 1 - 1 = 0 \neq n \in N$

$\therefore$ The element 1 in the codomain set has no pre-image in domain set

Hence g is not surjective.

2nd part :→ Let $S = \{1,2,3\}$ and $T = \{1,2\}$ be two finite set and $f: S \to T$ be a mapping defined by $f(1) = 2$, $f(2) = 2$, $f(3) = 1$ which shows that f is not injective but surjective

Again, Let $A = \{1,2\}$ and $B = \{1,2,3\}$ be two finite sets and $g: A \to B$ a mapping defined as $g(1) = 1$, $g(2) = 2$; which shows that g is injective but not surjective.

This shows that we can construct similar maps on replacing N by a finite set.

5 b) Find the range of the function $f: R \to R$ defined by,

$$f(x) = \left.\begin{array}{ll} x^2 + 1 & \text{if } x \geq 0 \\ = 1 - x & \text{if } x < 0 \end{array}\right\},$$

R being the set of reals.

$\Rightarrow$ Range of $f = \{f(x) : x \in R\}$

$= \{f(x) : x \in R \text{ and } x \geq 0\} \cup \{f(x) : x \in R \text{ and } x < 0\}$

$= [1, \infty) \cup [1, \infty)$

$= [1, \infty)$.

6. (a) construct a binary relation on a set which is reflexive and symmetric but not transitive.

⇒ Let $A = \{1, 2, 3\}$.

Let $\mathbb{R}$ be a binary relation defined on the set A by,

$\mathbb{R} = \{(1,1), (2,2), (3,3), (1,2), (2,1), (2,3), (3,2)\}$.

Here $a \mathbb{R} a$ holds, $\forall a \in A$

$\therefore \mathbb{R}$ is reflexive.

$a R b \Rightarrow b \mathbb{R} a \quad \forall a, b \in A$

$\therefore R$ is symmetric.

but $(1,2) \in \mathbb{R}$ and $(2,3) \in \mathbb{R}$ but $(1,3) \notin \mathbb{R}$

$\therefore \mathbb{R}$ is not transitive.

b) show that the mapping $g: \mathbb{C} \to \mathbb{C} - \{0\}$, defined by $g(z) = \exp z$ is surjective but not injective.

⇒ $g: \mathbb{C} \to \mathbb{C} - \{0\}$; $g(z) = e^z$

Let $z_1, z_2 \in \mathbb{C}$ where $z_1 = 2\pi i$, $z_2 = 0$

Then $g(z_1) = e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1$.

$g(z_2) = e^0 = \cos 0 + i \sin 0 = 1$

$\therefore z_1 \neq z_2 \Rightarrow$ but $g(z_1) = g(z_2)$

$\therefore g(z) = \exp z$ is not injective.

Again, $g(\mathbb{C}) = \mathbb{C} - \{0\}$,

$\therefore g(z) = e^z$ is surjective.